

PROPER ISOMETRIC ACTIONS

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In this short note we use a theorem by Helgason to give an easy proof of two results on proper isometric actions (theorems 4 and 5).

Throughout this paper we assume that M is a connected and complete Riemannian manifold. We denote by $I(M)$ the full isometry group of M . It is a well known fact that $I(M)$ is a Lie group [3].

Let G be a Lie group. An action $\varphi : G \times M \rightarrow M$ is said to be an *isometric action* if for each $g \in G$ the map $\varphi_g : M \rightarrow M, p \mapsto g(p)$, is an isometry of M . It is customary to restrict to effective actions so that G can be considered a subgroup of $I(M)$ (not necessarily closed).

Our basic tool is the following theorem, which was proved in [1, p. 167] (see also [4]).

Theorem 1. *Let M be a complete, connected Riemannian manifold and let $\{g_n\} \subset I(M)$ be a sequence such that $\{g_n(p)\}$ is bounded in M for some $p \in M$. Then, there exists an isometry $g \in I(M)$ and a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $g_{n_k} \rightarrow g$.*

An action of a Lie group G on a manifold M is called *proper* [2] if one (and hence all) of the following conditions is satisfied:

- (i) The map $G \times M \rightarrow M \times M, (g, p) \mapsto (g(p), p)$ is proper.
- (ii) Given compact subsets K and L of M , the set $\{g \in G : g(K) \cap L \neq \emptyset\}$ is compact.
- (iii) For any two points $p, q \in M$, there exist open neighborhoods U_p and V_q of p and q respectively such that $\{g \in G : g(U_p) \cap V_q \neq \emptyset\}$ is relatively compact.
- (iv) For any sequences $\{g_n\} \subset G$ and $\{p_n\} \subset M$, if $g_n(p_n) \rightarrow q$ and $p_n \rightarrow p$ then $\{g_n\}$ has a convergent subsequence.

For isometric actions condition (iv) can be simplified:

Lemma 2. *Let $G \subset I(M)$. Then G acts properly on M if and only if for any sequence $\{g_n\} \subset G$ such that $\{g_n(p)\}$ is bounded for some $p \in M$, there exists a convergent subsequence of $\{g_n\}$ in G .*

Proof. The “if” part is obvious because any bounded sequence of points of M has a convergent subsequence. Conversely, let $\{g_n\} \subset G$ and $\{p_n\} \subset M$ be sequences such that $g_n(p_n) \rightarrow q$ and $p_n \rightarrow p$. Since $0 \leq d(g_n(p), q) \leq d(g_n(p), g_n(p_n)) + d(g_n(p_n), q) = d(p, p_n) + d(g_n(p_n), q)$ we have $g_n(p) \rightarrow q$. In particular, $\{g_n(p)\}$ is bounded. By hypothesis there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $g_{n_k} \rightarrow g \in G$. Hence G acts properly on M . \square

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Corollary 3. *A subgroup $G \subset I(M)$ acts properly on M if and only if for any sequence $\{g_n\} \subset G$ and any $p \in M$, $g_n(p) \rightarrow q$ implies that $\{g_n\}$ has a convergent subsequence in G .*

It is well known that a closed subgroup of the isometry group of a Riemannian manifold acts properly on that manifold. The converse fact seems to be a folklore theorem, but the author has not found a proof in the literature. We give a simple proof using Theorem 1.

Theorem 4. *Let $G \subset I(M)$ be a subgroup. Then, G acts properly on M if and only if G is closed in $I(M)$.*

Proof. First assume that G acts properly on M . Let $\{g_n\}$ be a sequence with $g_n \rightarrow g \in I(M)$. Let $p \in M$. Then, $g_n(p) \rightarrow g(p)$, and so by Corollary 3 there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \rightarrow h \in G$. The uniqueness of limit implies $g = h \in G$, so G is closed.

Conversely, assume that G is closed. Let $\{g_n\} \subset G$ be a sequence with $g_n(p) \rightarrow q$. By Theorem 1, there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \rightarrow g \in I(M)$. Since G is closed it follows that $g \in G$ and thus G acts properly by Corollary 3. \square

The following result states that, up to orbit equivalence, isometric actions with closed orbits correspond to proper isometric actions.

Theorem 5. *Let $G \subset I(M)$ be a subgroup. Then, the orbits of G are closed if and only if the action of G is orbit equivalent to the action of the closure of G in $I(M)$.*

Proof. Let \bar{G} denote the closure of G in $I(M)$.

Assume that the orbits of G are closed. Let $p \in M$. Obviously, $G \cdot p \subset \bar{G} \cdot p$. Let $q \in \bar{G} \cdot p$ and write $q = g(p)$ with $g \in \bar{G}$. Take $\{g_n\} \subset G$ such that $g_n \rightarrow g$. Then, $g_n(p) \rightarrow g(p) \in G \cdot p$ because $G \cdot p$ is closed. By uniqueness of limit we have $q = g(p) \in G \cdot p$.

Conversely, assume that the action of G is orbit equivalent to the action of \bar{G} . Let $p \in M$ and take $\{g_n\} \subset G$ with $g_n(p) \rightarrow q \in M$. By Theorem 1 there is a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \rightarrow g \in \bar{G}$. Hence $g_{n_k}(p) \rightarrow g(p)$. By uniqueness of limit $q = g(p) \in \bar{G} \cdot p = G \cdot p$ so $G \cdot p$ is closed. \square

Corollary 6. *The orbits of an isometric action are closed if and only if the action is orbit equivalent to a proper isometric action.*

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